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Local and Global Algebraic Structures in General Relativity

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The extent to which the well-known pointwise algebraic canonical forms used for the energy-momentum tensor, the Weyl tensor, etc., can be regarded as smooth relations over some open subset of (possibly the whole of) space-time is investigated.

1. INTRODUCTION

The algebraic structure of symmetric and skew-symmetric second-order tensors and fourth-order tensors with Weyl symmetry at a point in space-time is well known [see, e.g., Petrov (1969), Pirani (1957), Bel (1962), Penrose (1960), Sachs (1961), Plebanski (1964), Hall (1976), and Synge (1956); for reviews see Hall (1984) and Kramer *et al.*, (1980)]. However, the canonical forms obtained at p are usually assumed to hold smoothly in some neighborhood of p or even over the whole space-time manifold. These assumptions need justification and may be false without further conditions. The purpose of this paper is to investigate these problems more rigorously.

Throughout, M will denote a (connected, smooth) space-time manifold and g a smooth Lorentz metric on M. If $p \in M$, T_pM denotes the tangent space to M at p. An *m-dimensional distribution* on M is a map Ω which associates with each $p \in M$ an m-dimensional subspace of T_pM and such a map is called *smooth* if for each $p \in M$ there is an open neighborhood U of p in M and m smooth vector fields on U whose values at each $q \in U$ span $\Omega(q)$. Using the metric g and given an m-dimensional distribution Ω on M, one can define, in an obvious way, the *orthogonal complement* of Ω, which is then a $(4\text{-}m)$-dimensional distribution on M. If A is a second-order

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(real) mixed tensor at p, the *range* of A is the subspace of T_pM given in local coordinates by $\{A^a_b v^b: v^a \in T_pM\}$ and the dimension of this subspace is the *rank* of A at p. The *kernel* of A at p is the subspace $\{v^a: A^a_b v^b = 0\}$ of T_pM and its dimension is $4-$(rank of A at p). In dealing with the algebraic structure of mixed second-order tensors at $p \in M$ (that is, the general solution of the eigenvector–eigenvalue problem $A^a_b v^b = \lambda v^a$ for the real or complex vector v and $\lambda \in \mathbb{C}$), use will be made of the Segre notation for the corresponding Jordan form (modified in a well-known way to allow for conjugate pairs of complex eigenvalues). An eigenvalue of such a tensor, whether corresponding to a simple or a nonsimple elementary divisor, is called *nondegenerate* (respectively of degeneracy r, $r > 1$) if the corresponding eigenspace is 1-dimensional (respectively, r-dimensional). The *algebraic type* of such a tensor will be regarded as completely specified if its Segre symbol is given together with any eigenvalue degeneracies.

The standard sets $\mathbb{R}^n\mathbb{C}$, $M_n\mathbb{R}$, etc., will always be assumed to have their usual real manifold structures.

2. PRELIMINARY RESULTS

This section discusses the smoothness of the eigenvalues of smooth, second-order mixed tensors on M and the smoothness of certain distributions associated with these tensors. Many of these results probably exist in the literature, but are not always easily found, at least in the form required here. Most of the proofs are straightforward and so the discussion will be brief. Although the approach has the space-time M in mind, the results are mostly applicable to higher-dimensional manifolds.

(i) Let U denote the open submanifold of nonsingular members of $M_n\mathbb{R}$ and let $A \in U$ and $y \in \mathbb{R}^n$. The unique solution $x \in \mathbb{R}^n$ of the system of equations $Ax = y$ depends smoothly on A and y in the sense that the map $(A, y) \to x$ is a smooth map $U \times \mathbb{R}^n \to \mathbb{R}^n$. As a consequence, if A is a smooth, mixed, second-order tensor of rank 4 and Y a smooth vector field each defined on some open subset $V \subseteq M$, there is a unique smooth vector field X on V which satisfies $A^a_b X^b = Y^a$ in any chart domain in V.

The second part follows from the first by noting that A may be regarded as a smooth map $W \to U \subseteq M_n\mathbb{R}$ and Y as a smooth map $W \to \mathbb{R}^n$ for any chart domain W in V.

Now let P_n be the set of polynomials of degree $\leq n$ and with real coefficients. Then P_n can be identified with the manifold $\mathbb{R}^{n+1}$ by the map

$$P \equiv a_n x^n + \cdots + a_1 x + a_0 \to (a_0, a_1, \ldots, a_n)$$

Then let $\lambda \in \mathbb{C}$ be a simple root of $P \in P_n$ in the sense that $P(\lambda) = 0$, $P'(\lambda) \neq 0$.

Define smooth maps f and g by

$$f\colon\quad \mathbb{C}\times\mathbb{R}^{n+1}\to\mathbb{C}\qquad f(z, a_0, \ldots, a_n)=a_n z^n+\cdots+a_1 z+a_0$$

$$g\colon\quad \mathbb{R}^2\times\mathbb{R}^{n+1}\to\mathbb{R}^2\quad g(x, y, a_0, \ldots, a_n)=(a, b)$$

where $a+ib=f(x+iy, a_0, \ldots, a_n)$. Now the Cauchy-Riemann equations show that the Jacobian $\partial(a, b)/\partial(x, y)=|P'(\lambda)|^2\neq 0$ when evaluated at the point in $\mathbb{R}^2\times\mathbb{R}^{n+1}$ determined by λ and P and so the implicit function theorem reveals the existence of a smooth map h from an open neighborhood U of P to $\mathbb{C}$ such that $h(P)=\lambda$ and if $Q\in U$, $h(Q)$ is a root of Q. The following result is then obtained.

(ii) The simple roots of a polynomial depend smoothly on the polynomial coefficients.

The next result is a special case of a theorem in Dieudonné (1969, p. 248).

(iii) Let $V\subseteq\mathbb{C}$ be open and let $f\colon V\times M_n\mathbb{R}\to\mathbb{C}$ be the smooth map given by $f(z, A)=\chi(z)$, where χ is the characteristic polynomial of A. It then follows that if $V'\subseteq V$ is open and has compact closure $\bar{V}'\subseteq V$ and if $A_0\in M_n\mathbb{R}$ is such that no zero of $f(z, A_0)$ lies on the boundary of V', then there exists an open neighborhood W of A_0 in $M_n\mathbb{R}$ such that (a) for any $A\in W$, $f(z, A)$ has no zeros on the boundary of V', and (b) for any $A\in W$, the sum of the orders of the zeros of $f(z, A)$ in V' is independent of A.

(iv) Let $\mathscr{A}$ be a subset of $M_n\mathbb{R}$ consisting of matrices all of which have the same algebraic type in the sense defined in Section 1. Then for any $A\in\mathscr{A}$ there is an open neighborhood U of A in $M_n\mathbb{R}$ and smooth maps $h_1, \ldots, h_k\colon U\to\mathbb{C}$ such that for $B\in\mathscr{A}\cap U$, $h_1(B), \ldots, h_k(B)$ are the distinct eigenvalues of B.

To see this, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A and choose disjoint neighborhoods $U_1, \ldots, U_k$ of $\lambda_1, \ldots, \lambda_k$ in $\mathbb{C}$. If λ_i has multiplicity m as a root of the characteristic polynomial $\chi(A)$ of A, one can arrange, by reducing the size of U_i, if necessary, that λ_i is the only root of the $(m-1)$th derivative $\chi^{(m-1)}(A)$ contained in $\bar{U}_i$. By (iii) there exists an open neighborhood W of A in $M_n\mathbb{R}$ such that if $B\in W$, all eigenvalues of B are contained in $\bigcup_{i=1}^{k} U_i$ and such that only one root of $\chi^{(m-1)}(B)$ is contained in U_i. This is a simple root of $\chi^{(m-1)}(B)$ and so depends smoothly on B by (ii). The smooth function thus obtained is h_i.

By regarding a smooth, mixed second-order tensor A on M as a smooth map $V\to M_n\mathbb{R}$ for any coordinate domain V of M and given that A has the same algebraic type at each $p\in M$, then (iv) above shows that *any eigenvalue of A may be regarded as locally smooth on M.* If also A has some real eigenvalues, the distinct real eigenvalues of A may be ordered by size in the usual way in $\mathbb{R}$ and then each gives rise to a *global* real-valued function on M which is smooth by the local smoothness results of the previous sentence. One thus has the following result.

(v) Let A be a smooth, mixed second-order tensor on M with the same algebraic type at each $p \in M$. Then all eigenvalues of A are locally smooth in the sense that for each $p \in M$ there is an open neighborhood U of p such that the eigenvalues of A may be regarded as smooth functions on U. Any real eigenvalue of A may be regarded as a global smooth function on M.

Now the smoothness of the eigendirections of the tensor A can be considered.

(vi) Let A be a smooth, mixed second-order tensor on M which has the same rank r at each $p \in M$. Then the range of A is an r-dimensional, smooth distribution on M.

This follows easily, since if $p \in M$ and U a coordinate neighborhood of p, let $X_1, \ldots, X_4$ be the standard coordinate vector fields on U. Then r of the smooth vector fields with components $A^a_b X^b_1, \ldots, A^a_b X^b_4$ are independent at p and hence in some open neighborhood of p and span the range of A there.

(vii) Let Ω be a smooth distribution on M of dimension 1, 2, or 3. Then the orthogonal complement of Ω is a smooth distribution on M.

To establish this, note that if Ω is 3-dimensional and U is any connected chart domain of M containing smooth vector fields X, Y, and Z spanning Ω in U, then $\eta^a_{\ bcd} X^b Y^c Z^d$ (where η is the alternating symbol in U)[3] is a smooth vector field on U spanning the orthogonal complement of Ω. If Ω is 2-dimensional, then about any $p \in M$ there is a connected coordinate neighborhood U of p and smooth vector fields X and Y on U spanning Ω. Thus, the bivector on U with components $F^{ab} = 2X^{[a}Y^{b]}$ is smooth on U and hence so is the dual $\overset{*}{F}{}^{ab}$ of F^{ab}. Now, $\overset{*}{F}{}^a_b$ has rank 2 at each point of U and so, by (vi), its range is a 2-dimensional smooth distribution on U. Since $p \in M$ was arbitrary, this construction yields the orthogonal complement of Ω and shows that it is smooth. If Ω is 1-dimensional, then about any $p \in M$ there is a connected coordinate domain U and vector fields K, R, S on U such that K spans Ω in U and K, R, and S are independent at every $p \in U$. Then the bivectors $K^{[a}R^{b]}$ and $K^{[a}S^{b]}$ are smooth and independent at each $p \in U$ and hence so are their duals. The two 2-dimensional smooth distributions these duals give rise to, by (vi), then generate in an obvious way a 3-dimensional smooth distribution on M which is the orthogonal complement of Ω.

(viii) Let A be any smooth, mixed second-order tensor on M whose algebraic type is the same at each $p \in M$. Then the (real) eigenspaces

[3]The usual sign change due to the two orientations on U is assumed incorporated into η.

corresponding to real eigenvalues of A give rise to smooth distributions on M and hence may be spanned locally by smooth eigenvector fields of A.

To prove this, note that the eigenvalues of A may be regarded as locally smooth functions on M by (v). Let $p \in M$ and choose a coordinate neighborhood U of p such that the eigenvalues of A are smooth functions on U. Let α be a real eigenvalue of A with degeneracy r, $1 \leq r \leq 3$ (and r is constant on U because of the assumption regarding the algebraic type of A) and let A have components A^a_b on U. Consider the smooth, mixed, second-order tensor with components $B^a_b \equiv A^a_b - \alpha\delta^a_b$ on U. This tensor has constant rank equal to $4-r$ on U, since at each $p \in U$ the α eigenspace of A is the kernel of B^a_b. Now consider the smooth, mixed, second-order tensor field defined on U by its having components ${}^{\mathrm{T}}B^a_b$. This tensor has constant rank $4-r$ on U and its range gives rise to a $(4-r)$-dimensional smooth distribution on U by (vi). The orthogonal complement of this distribution is then an r-dimensional smooth distribution on U by (vii), which coincides with the α eigenspace of A at each $p \in U$. The case $r=4$ is trivial because then $A^a_b = \alpha\delta^a_b$ on U.

(ix) If Ω is a smooth null distribution of dimension 2 or 3 on M [that is, $\Omega(p)$ is a null 2-space (or a null 3-space) at each $p \in M$], then one of its local smooth spanning vector fields may be chosen null [that is, as spanning the unique null direction in $\Omega(p)$ at each $p \in M$]. If Ω is a smooth 2-dimensional timelike distribution on M, then it may be locally spanned by two smooth null vector fields.

To see this, note that, for example, if Ω is null and 2-dimensional and has local spanning fields X and Y in some coordinate domain U of M, then if neither X nor Y is null in U, the required smooth vector is the vector field L on U uniquely determined by the system of equations $X_aL^a = Y_aL^a = Z_aL^a = 0$, $V_aL^a = 1$, where Z is any smooth vector field lying in the smooth distribution orthogonal to Ω which is nowhere null in U, and V is a smooth, timelike vector field on U. The domain U can always be chosen such that V and Z are defined in U and the result follows from (i) above. The proof when Ω is 3-dimensional and null or 2-dimensional and timelike is similar.

3. APPLICATIONS

3.1. Symmetric Second-Order Tensors

At any $p \in M$ a second-order symmetric tensor with components T_{ab} in some coordinate domain of p poses the eigenvector–eigenvalue problem $T^a_bk^b = \alpha k^a$ and it is known (Plebanski, 1964; Hall, 1976, 1984) that either T has a conjugate pair of complex eigenvalues together with two real ones or else all the eigenvalues are real. The former case is labeled as Segre type $\{z\bar{z}11\}$ [or $\{z\bar{z}(11)\}$ if the two real eigenvalues are equal] and in this case T is diagonalizable over $\mathbb{C}$ and all the eigenvalues are simple. In the latter case one may cast T into Jordan canonical form and the Segre types consistent with the Lorentz signature of the metric g (since T must satisfy $g_{ac}T^c_b = g_{bc}T^c_a$) are the Segre types $\{1, 111\}$, $\{211\}$, and $\{31\}$ and their degeneracies, which are denoted in the usual way by the use of round brackets. The type $\{1, 111\}$ and its degeneracies are diagonalizable over $\mathbb{R}$ and are the only types possessing a timelike eigendirection, which, while not necessarily unique (depending on the degeneracies), corresponds to a unique eigenvalue which is signified in the $\{1, 111\}$ notation by the first digit and is separated from the others by a comma.

If, at $p \in M$, T has respective Segre types $\{z\bar{z}11\}$, $\{1, 111\}$, $\{211\}$, and $\{31\}$ (or their degeneracies), then it may be written at p in the respective canonical forms

$$T_{ab} = 2\rho_1 l_{(a}n_{b)} + \rho_2(l_a l_b - n_a n_b) + \rho_3 x_a x_b + \rho_4 y_a y_b \qquad (\rho_2 \neq 0) \tag{1}$$

$$T_{ab} = 2\rho_1 l_{(a}n_{b)} + \rho_2(l_a l_b + n_a n_b) + \rho_3 x_a x_b + \rho_4 y_a y_b \tag{2}$$

$$T_{ab} = 2\rho_1 l_{(a}n_{b)} + \lambda l_a l_b + \rho_2 x_a x_b + \rho_3 y_a y_b \qquad (\lambda \neq 0) \tag{3}$$

$$T_{ab} = 2\rho_1 l_{(a}n_{b)} + 2\sigma l_{(a}x_{b)} + \rho_1 x_a x_b + \rho_2 y_a y_b \qquad (\sigma \neq 0) \tag{4}$$

where (l, n, x, y) is a real null tetrad at p whose only nonvanishing inner products are $l^a n_a = x^a x_a = y^a y_a = 1$ and λ, σ, and the ρ's are real numbers. An alternative form for (2) is

$$T_{ab} = \lambda_1 x_a x_b + \lambda_2 y_a y_b + \lambda_3 z_a z_b - \lambda_4 u_a u_b \tag{5}$$

where (u, x, y, z) is a pseudo-orthonormal tetrad whose only nonvanishing inner products are $-u^a u_a = x^a x_a = y^a y_a = z^a z_a = 1$ and the λ's are real numbers. In (1) the eigenvalues are $\rho_1 \pm i\rho_2$, ρ_3, ρ_4 with corresponding eigenvectors $l \pm in$, x, y. In (2) the eigenvalues are $\rho_1 \pm \rho_2$, ρ_3, ρ_4 [λ_1, λ_2, λ_3, λ_4 in (5)] with corresponding eigenvectors $l \pm n$, x, y (x, y, z, u). In (3) the eigenvalues are ρ_1, ρ_2, ρ_3 with corresponding eigenvectors l, x, y. In (4) the eigenvalues are ρ_1, ρ_2 with corresponding eigenvectors l, y. One now has the following result.

(x) If T is a smooth, second-order symmetric tensor on M whose algebraic type is the same at each point of M (in the sense given in Section 1), the eigenvalues and eigenvectors of T may be regarded as locally smooth [and the real eigenvalues in (1)-(5) as globally smooth] on M. In particular, the canonical forms (1)-(5) may be regarded as holding in a coordinate domain of any $p \in M$ with the ρ's and λ's regarded as smooth functions on this domain and the tetrad members in each case regarded as smooth vector fields in this domain. Further, one may always set $\lambda = \pm 1$ and $\sigma = 1$ in (3) and (4) over this domain without affecting any of the above results.

The proof can largely be gathered from the results of the previous section. For example, if T has Segre type $\{1, 111\}$ everywhere with no degeneracies, then one obtains locally smooth eigenvalues and eigenvectors from results (v) and (viii), the normalization required to achieve equation (5) being obviously smooth. Equation (1) is then readily obtained from equation (5). When T has Segre type $\{211\}$ everywhere with no degeneracies, results (v) and (viii) again establish the local smoothness of l, x, y, and the ρ's in (3). The existence of the smooth distribution orthogonal to that spanned by x and y then shows, using (ix), that n is locally smooth. The local smoothness of λ then follows and hence the locally smooth adjustments required to achieve $\lambda = \pm 1$. In the $\{31\}$ case [equation (4)], l, y, and the ρ's are locally smooth by (v) and (viii) and the completeness relation $g_{ab} = 2l_{(a}n_{b)} + x_a x_b + y_a y_b$ then shows that $\sigma l_{(a}x_{b)}$ is locally smooth, since g and $T - \rho_1 g$ are. The local smoothness of l and the components $\sigma l_{(a}x_{b)}$ then show that x and σ may be chosen locally smooth, and the locally smooth scaling to achieve $\sigma = 1$ and the local smoothness of n then follow. In the $\{z\bar{z}11\}$ case the presence of complex eigenvalues necessitates a slightly different approach. However, one can still use (v) to see that ρ_3, ρ_4, and $\rho_1 \pm i\rho_2$ (and hence ρ_1 and ρ_2) are locally smooth and (viii) to see that x and y are locally smooth. The completeness relation then shows that $l_{(a}n_{b)}$ is locally smooth and hence so is $(l_a l_b - n_a n_b)$, since T is. From this and from (vii) and (ix) one can deduce that l and n may be chosen locally smooth. For those algebraic types described by Segre types with degeneracies the proofs are similar except that one has the extra but straightforward task of smoothly "orthonormalizing" within each eigenspace.

3.2. Bivectors

It is briefly remarked here that if F is a smooth, second-order, skew-symmetric tensor (bivector) on M which is of the same type (spacelike, timelike, null, or nonsimple) at each $p \in M$, then it is of the same algebraic type at each $p \in M$ in the sense described in Section 1 [the respective Segre types being $\{(11)z\bar{z}\}$, $\{11(11)\}$, $\{(31)\}$, or $\{11z\bar{z}\}$ with no further degeneracies permitted]. The blade of F (when F is simple) and the canonical pair of blades of F (when F is nonsimple) give rise to smooth distributions on M and the usual local canonical forms for F may then be regarded as written in terms of locally smooth functions and vector fields in an obvious way as follows from the results of Section 2.

3.3. The Weyl Tensor

Results similar to those obtained above for the energy-momentum tensor hold also for the (smooth) Weyl tensor when it is decomposed into its canonical Petrov forms. To see this, one could proceed in a way similar to that given above by considering the algebraic structure of the Weyl tensor components regarded in the usual way as a smooth, symmetric, 6×6 matrix-valued function on any chart of M. However, it is more convenient to consider the complex self-dual Weyl tensor regarded in the usual way as a smooth, complex, symmetric, 3×3 matrix-valued function on any chart of M. Unfortunately, the basic results in Section 2 concerned real matrices, but it turns out that the key results (ii)–(v) can be modified to deal with the complex case and that (v) still holds if A is any smooth, complex matrix on M having the same algebraic type at each $p \in M$ in the sense that the (complex) eigenvalues of A can be chosen locally smooth functions in some neighborhood of any $p \in M$. Thus, the Weyl eigenvalues (Petrov scalars) can be chosen locally smooth in this sense and one proceeds by showing how the Weyl eigenbivectors of the complex self-dual Weyl tensor can be chosen locally smooth by techniques closely related to those in Section 2. For this reason only a summary of the main results is given.

If the Petrov type is I at every $p \in M$ so that there are three distinct complex Petrov scalars at each $p \in M$, then these scalars and their corresponding complex eigenbivectors can be chosen locally smooth on M. The (real) canonical blades determined by the real and imaginary parts of these eigenbivectors determine three spacelike and three timelike smooth 2-dimensional distributions on M whose intersections determine three spacelike and one timelike 1-dimensional distributions on M. These 1-dimensional distributions fix, up to signs, the canonical Petrov tetrad at each $p \in M$ and in this sense the canonical Petrov tetrads can be regarded as four local smooth vector fields on M. From these local smooth vector fields one can construct local smooth null tetrads in an obvious sense and express the complex self-dual Weyl tensor in a locally smooth canonical form of the type used by Bel (1962), Sachs (1961), and others, and since the construction of the so-called Debever–Penrose directions involves finding simple roots of a fourth-order polynomial equation, one also has four smooth null 1-dimensional distributions determined on M spanned at each $p \in M$ by the Debever–Penrose directions. In this sense one may choose in some neighborhood of each point $p \in M$ four distinct smooth null Debever–Penrose vector fields.

Similar arguments hold for the other Petrov types provided the Petrov type is the same at each $p \in M$. Thus, the Petrov scalars (which are all identically zero for types III and N) and associated eigenbivectors and repeated and nonrepeated Debever–Penrose vectors may all be regarded as locally smooth on M. The canonical forms for each of these types as given, for example, in Bel (1962) and Sachs (1961), may also be regarded as locally smooth in the sense of the previous paragraph.

4. GLOBAL CONSIDERATIONS

Suppose a smooth, symmetric, second-order tensor on M has the same algebraic type at each $p \in M$ and has all its eigenvalues real. Then result (v) shows that the eigenvalues can be regarded as global, smooth functions on M. However, one may not be able to assume the existence of global, smooth eigenvector fields. The following result shows that under certain conditions global smooth eigenvector fields exist.

(xi) Let M be a *simply connected* space-time and T a smooth, symmetric, second-order tensor on M whose algebraic type is the same at each $p \in M$ and whose eigenvalues, whether simple or nonsimple, are all real and nondegenerate. Then there exist global smooth vector fields on M which span the eigendirections of T at each $p \in M$. In particular, if T is of Segre type $\{1, 111\}$ at each $p \in M$, then[4] M admits four global, unit smooth vector fields, one timelike and three spacelike, which span the eigendirections of T at each $p \in M$ and, as a consequence, M is parallelizable. M is also parallelizable if T is of Segre type $\{211\}$ at each $p \in M$.

To see this, note that for each (global, smooth) eigenvalue the corresponding eigenspace gives rise to a smooth 1-dimensional distribution on M which is of the same nature (timelike, spacelike, or null) at each point of M. Now, since M is simply connected, it admits a global, smooth, nowhere zero timelike vector field u. Thus, if a certain eigenspace is timelike or null, it gives rise to a global, smooth eigenvector field on M by insisting on the normalizing condition $g(u, k) = 1$ for each of the local, smooth eigenvector fields k. If the eigenspace is spacelike, one first insists on the normalization $g(k, k) = 1$ for the local, smooth eigenvector fields. If with this restriction one cannot choose k globally and smoothly on M, one can always find a twofold covering space of M by a standard argument and this contradicts the simply connectedness of M. That M is parallelizable in the $\{1, 111\}$ case is clear and in the $\{211\}$ case follows from the existence of the vector field u above and the three eigenvector fields.

It follows that if a space-time M admits a global, smooth, second-order symmetric tensor field of Segre type $\{1, 111\}$ at each $p \in M$, then M admits an n-fold covering space which is parallelizable (and it is easily seen that $n \leq 2^4 = 16$). This result restricts the topology of a manifold on which the energy-momentum tensor has Segre type $\{1, 111\}$ everywhere. In particular, it cannot be both simply connected and nonparallelizable.

A similar discussion can be given for the Weyl tensor and leads to the following result.

[4]A similar result is true for the eigendirections arising from the (nondegenerate) real eigenvalues in the $\{z\bar{z}11\}$ case.

(xii) Let M be a simply connected space-time and suppose that the Petrov type is the same at each $p \in M$. Then there exist global, null vector fields on M which span the Debever-Penrose directions at each $p \in M$. In particular, if the Weyl tensor is type I at each $p \in M$, there exist four global, null vector fields on M which span the Debever-Penrose directions at each $p \in M$ and four mutually orthogonal, global, smooth unit vector fields on M, one timelike and three spacelike, which span the Petrov tetrads at each $p \in M$. As a consequence, M is parallelizable in this case.

It follows that if M is a space-time whose Petrov type is I at each $p \in M$, then M admits an n-fold covering space which is parallelizable (and, in fact, $n \leq 2 \times 2^3 \times 3! = 96$).

5. CONCLUDING REMARKS

The smoothness results concerning eigenvectors given in (viii), (x), and elsewhere required the assumption of constancy of algebraic type, whereas the global results in (xi) and (xii) also required M to be simply connected. These results may fail if these assumptions are dropped, as will be briefly demonstrated here with a second-order tensor on a 2-dimensional manifold (and which is readily extended to higher-order examples). Let $M' = \mathbb{R}^2$ with the usual Euclidean metric and let A be a global, smooth, second-order symmetric tensor defined on M' in the usual global Cartesian coordinates by the component form

$$A_{ab}(x, y) = e^{-1/r^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \quad (x^2+y^2 \neq 0), \qquad A_{ab}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

where r and θ are the usual polar functions on $\mathbb{R}^2 \setminus \{(0,0)\}$. Then A is diagonalizable everywhere having Segre type $\{1,1\}$ with eigenvalues $\pm e^{-1/r^2}$ and corresponding eigenvectors $(\cos(\theta/2), -\sin(\theta/2))$ and $(\sin(\theta/2), \cos(\theta/2))$ if $r \neq 0$ and type $\{(11)\}$ with degenerate eigenvalue zero at $(0,0)$. Thus, a change of algebraic type occurs at the origin. It is then clear that although local eigenvector fields exist in a neighborhood of any point $(x, y) \neq (0,0)$, the same is not true of the origin. For, suppose there is an open ε-ball $(\varepsilon > 0)$ about $(0,0)$ in which two independent smooth eigenvector fields are defined. Then they may be assumed normalized with respect to the Euclidean metric and such that they have components $(1,0)$ and $(0,1)$ at $P = (\delta, 0) \in U$ $(0 < \delta < \varepsilon)$. Now consider the circular path $P \to P$ given by $t \to (\delta\cos t, \delta\sin t)$. The smooth extensions of the eigenvectors at P along this curve have components $(\cos(t/2), -\sin(t/2))$ and $(\sin(t/2), \cos(t/2))$ and are easily seen to have reversed their signs, compared with the original eigenvectors at P, on arrival back at P. Thus, one obtains a contradiction. Alternatively, one could obtain a contradiction by supposing smooth unit eigenvector fields exist in U and smoothly extending the resulting eigenvectors at a point $Q \neq (0, 0)$ with polar angle θ to $(0, 0)$ along the straight line of constant θ in U. The eigenvectors so obtained at $(0, 0)$ depend on θ and hence no continuous extension of these eigenvector fields to $(0, 0)$ is possible.

Now let $M'' = \mathbb{R}^2 \backslash \{(0, 0)\}$ and take A to be the above tensor restricted to M''. The above shows that the algebraic type of A is the same at each point of the non-simply connected manifold M'' but the first contradiction argument given above shows that global, smooth eigenvector fields of A do not exist on M''.

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